The Relation between Counter-party Default and Interest Rate Volatility, and Its Impact on the Credit Risk of Interest Rate Derivatives

Geoffrey R. Harris
Federal Reserve Bank of Chicago

Tao L. Wu*
Illinois Institute of Technology, Stuart School of Business, 565 West Adams Street, Suite 458, Chicago, IL 60661, USA; email: tw33_99@yahoo.com

Jiarui Yang
FactSet Research Systems

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ABSTRACT

We present a unified framework to study the impact of the correlation between interest rate volatility and counter-party default probability on the credit risk of collateralized interest rate derivatives contracts. When interest rates are volatile, counter-parties are potentially more likely to default. Large moves in interest rates accompanied with counter-party default may lead to losses on interest rate derivatives even if they are collateralized. An interest rate model with stochastic volatility and a reduced-form default model, in which the default probability is correlated with interest rate volatility, are proposed and estimated from market data. Then we analyze the effect of the correlation between interest rate volatility and a counter-party’s default probability on the credit risk of collateralized interest rate derivatives contracts. Our result shows that ignoring this correlation under-estimates the credit risk even with collateralized trades.
SECTION 1
INTRODUCTION

According to the Triennial and Semiannual survey by the Bank for International Settlements, by the end of 2010, there were $647.762 trillion outstanding gross notional of over-the-counter (OTC) derivatives, in which more than 77% (approximately $432.657 trillion) are interest rate contracts. Meanwhile, large financial institutions are linked by mutual derivatives agreements, which are usually collateralized to prevent huge losses caused by counter-party default. The collateral amount is typically adjusted based on the mark-to-market value of the derivative, together with some minimum threshold that is specified in the contract. The value of collateral is only posted to cover max(mark-to-market value of the derivative agreement - the minimum threshold, 0)\(^1\). Usually, the amount of the collateral is updated daily as the values of interest rate derivative agreements vary.\(^2\) A large interest rate movement in

\(^1\)In our simulation presented in Section 5, we assume that the minimum threshold is zero and the collateral is cash. When risky securities are posted as collateral, additional collateral amounts are required because the value of the collateral may decrease and then it will not be able to cover the loss caused by the default. See Abbate [1] for details of typical collateral requirements.

\(^2\)Here, we assume that financial institutions trade and clear their trades directly with one another. This happens in the bilateral OTC market. There are two more forms of the organization of derivatives market which are also popular and well known. One is defined as an OTC market with decentralized trading but with centralized clearing through a central counterparty (CCP). Another is an exchange-based market in which both trading and clearing are centralized through an exchange which is typically linked to a CCP. Introducing a CCP into the clearing process will decrease the default risk financial institutions incur. Before 2013, bilateral clearing was predominant. But the structure has been changing since the Dodd-Frank Wall Street Reform and Consumer Protection Act was announced in 2010, which requires many financial institutions to clear their vanilla derivatives through CCPs. Even though the existence of the clearing house reduces the risks the counterparty need to carry, the risk remains for the clearing house. In addition, a clearing house inadequately capitalized or requiring insufficient collaterals/margins may also pose counter-party risk. See also Abbate [1] for a discussion on how netting affects the collateral requirements of a portfolio of swaps.
a short period, which is more likely to occur when interest rate volatility is high, can potentially cause a huge credit loss arising from a shortfall between the value of the collateral and the value of the derivative contracts. For example, the counter-party may default and fail to post collateral one day. Then, it may still take several days to make sure that the counter-party has absolutely defaulted and then to unwind or replace the trades. During this period, the difference between the value of the derivative contracts when the portfolio is liquidated and the value of the collateral posted the day before the counter-party defaulted could be large. Actually, this will become even worse when big financial institutions are linked together with over-the-counter contracts. The default of one institution will cause losses in other institutions, which may cause themselves to default. Therefore, it is important to study the relationship between interest rate volatility and the default likelihood of financial institutions and how this relationship influences the credit risk of interest rate derivatives. We take three steps to achieve this goal:

1. Model interest rates and interest rate volatility to consistently match the observed yield curve and corresponding observations of prices of interest rate derivatives.

2. Quantify the correlation between interest rate volatility and default likelihood of different companies using information from both credit default swaps market and the interest rate market.

3. Evaluate this correlation’s influence on the cost of credit default for different companies.

We want to specify an interest rate model that captures the following characteristics of interest rates. First, the volatility of interest rates is stochastic and is driven by unspanned factors. Collin-Dufresne and Goldstein [14] document that un-
spanned stochastic volatility factors influence only interest rate derivatives and have little impact on bond yields. In [12, 15], Casasus, Collin-Dufresne, Goldstein, and Collin-Dufresne, Goldstein, Jones analyzed a fairly parsimonious model of unspanned stochastic volatility in the affine framework. Second, the interest rate volatility is hump-shaped as a function of maturity. By plotting the volatilities of zero-coupon bond yields against maturity over the period 1983-1998, Dai and Singleton [17] showed that the term structures of unconditional volatilities of bond yields are on average hump-shaped. In [27], Reno and Uboldi showed that a model with hump-shaped volatility could improve the performance of the interest rate model by not only incorporating unspanned volatility factors, but also improving the model in terms of yield curve estimation errors and cap pricing performance. Third, changes in interest rate volatility are correlated with changes in interest rates. For instance, both Andersen and Lund [3] and Ball, Torous [4] study the dynamics of the short-term interest rate. Their work show that the relative interest rate volatility is negatively correlated with interest rates while the absolute interest rate volatility is positively correlated with interest rates. Based on these considerations, we adopt the stochastic volatility multifactor model developed by Trolle and Schwartz [30], where they derived an analytical formula of the zero-coupon bond option in terms of a finite number of state variables, which enables us to price more complex interest rate derivatives.

As for quantifying the correlation between interest rate volatility and the credit spread, we start by choosing an appropriate credit model. In order to keep the hazard rate positive, we focus on the square-root process, proposed by Feller [20], which has been widely used in financial economics, appearing in term structure models such as the CIR model developed by Cox, Ingersoll, and Ross [16]. The popularity of this process is due to its positivity and tractability. In the square-root process, a state variable follows a diffusion in which both the drift and the diffusion coefficients are affine functions of the state variable itself. Multivariate extensions of the square-root
process have appeared in the term structure literature; see, for example, Duffie and Kan [19], Dai and Singleton [17]. Therefore, we can directly invoke the well developed pricing formulas and statistical properties to price the credit default swap (CDS) and simulate credit events. Another reason why we choose the CIR process for the credit model is because we want the variance of spreads to grow with their magnitude rather than being constant. Moreover, we assume that the interest rate is correlated with the credit spreads. Then we calibrate our credit model to credit default swaps and estimate the correlation between the random factors that drive interest rate volatility and default.

Last but not the least, we use the Monte Carlo method to simulate the default event of 15 financial institutions given the estimated interest rate model and dynamics of the hazard rate. Then, by adjusting the correlation parameter between interest rate volatility and default spread and repeating the above process, we can study the influence of this correlation on the loss and likelihood of a credit default event for different financial companies.

This paper joins a growing literature on the counter-party credit risk of derivatives contracts. See Brigo, Pallavicini and Papatheodorou [11], Morktter, Pleus and Westerfeld [25] for a discussion of recent papers on the counter-party credit risk of interest rate derivatives and credit derivatives, respectively. It is worth noting that the correlation between interest rate volatility and counter-party default on the credit exposure of interest swaps discussed in this article results in “wrong way risk”, a risk that arises when default risk of the counter-party and credit exposure of the transaction increase together. It is analogous to the wrong way risk in a credit default swap (CDS) resulted from the correlation between the reference entity defaulting and the seller of the CDS defaulting. While the latter has received significant attention, to our best knowledge, this article is the first to examine the former. In addition,
the counter-party risk in a CDS also differs from that in an interest rate swap due to asymmetry of future potential exposure and its large size as a percentage of the notional of the trade, although both can potentially cause a credit contagion in a crisis.

The rest of the paper is organized as follows: Section 2 describes the stochastic volatility term structure model of interest rates. Section 3 introduces the CIR credit risk model and demonstrates how to simulate multiple correlated CIR processes which will be used later in Section 6 to compute the counter-party risk. Section 4 discusses the pricing of credit default swaps. Model estimation and results are presented in Section 5. In Section 6, Monte Carlo results for credit models will be analyzed and an analysis of the impact of the correlation between interest-rate volatility and default probability on credit exposure will be carried out. Section ?? concludes the paper.
In this section, we introduce a tractable stochastic multi-factor model of the term structure of interest rate developed by Trolle and Schwartz [30] in 2008. The model discussed in their paper is based on the HJM model framework firstly introduced by Heath, Jarrow, and Morton [21] in 1992. Different from the original HJM model, the volatility term of the model studied by Trolle and Schwartz is stochastic and the random term involved is correlated with the random term of the interest rate model.

2.1 Model Specification

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, Q)\) be a complete filtered probability space and the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) generated by a standard N-dimensional \(Q\)-Wiener process \(W^Q(t)\). Then, we assume that the time-\(t\) instantaneous forward rate \(f(t, T)\) is driven by the following diffusion process (see more details in [30]):

\[
    df(t, T) = \mu_f(t, T)dt + \sum_{i=1}^{N} \sigma_{(f,i)}(t, T)\sqrt{v_i(t)}dW^Q_i(t),
\]

\[
    dv_i(t) = k_i(\theta_i - v_i(t))dt + \sigma_i\sqrt{v_i(t)}(\rho_i dW^Q_i(t) + \sqrt{1 - \rho_i^2}dZ^Q_i(t)) \quad i = 1, \ldots, N,
\]

where \(W^Q_i(t)\) and \(Z^Q_i(t)\) are independent standard Wiener Processes under risk-neutral measure \(Q\). \(\mu_f(t, T)\) and \(\sigma_{(f,i)}(t, T)\) are measurable functions mapping\(^3\) from \([0, T] \times \mathbb{R}\) to \(\mathbb{R}\). As an extension of the traditional HJM model, the volatility in the instantaneous forward rate model is stochastic and it is driven by N-dimensional random noise \([Z^Q_1(t), \ldots, Z^Q_N(t)]\) which is correlated to the noise term \([W^Q_1(t), \ldots, W^Q_N(t)]\) in the forward rate model. In other words, this model extends the traditional HJM

\(^3\)The diffusion coefficient function and drift coefficient function will be specified later, so that the dynamics defined by model (2.1) are Markovian and arbitrage free.
model by introducing stochastic volatility which is also correlated with the forward rate.

### 2.1.1 Basic Structure of the Diffusion and Drift Coefficients.

Because the dynamics given by (2.1) are not necessarily arbitrage-free in most cases, we need to specify the drift and diffusion coefficient functions in model (2.1) to rule out arbitrage. In 1992, Heath, Jarrow, and Morton (in [21]) proved that the absence of the arbitrage assumption implies the following relationship between the diffusion coefficient and the drift coefficient in model (2.1) (detailed proof can be found in [21]):

\[
\mu_f(t, T) = \sum_{i=1}^{N} v_i(t) \sigma_{(f,i)}(t, T) \int_{t}^{T} \sigma_{(f,i)}(t, u) du.
\]  

(2.3)

As seen from the above expression and model (2.1), the dynamics of instantaneous forward rate \(f(t, T)\) are determined by the initial forward rate curve \(f(0, T)\), the volatility function \(\sigma_{(f,i)}\), and the forward rate volatility \(v_i(t)\). The initial forward rate curve and the forward rate volatility are determined either by the dynamical model or the term structure. The forward rate volatility function has more flexibility to be specified in order to grant our model desirable properties. For example, Ritchken and Sankarasubramaniam, in [29], identifies conditions on the volatility structure of forward rates that permit the dynamics of the term structure to be represented by a two-dimensional state variable Markov process. Bhar and Chiarella, in [5], studied a class of volatility functions for the forward rate process, which allows the bond price dynamics in the HJM framework to be reduced to a finite dimensional Markovian system, and so on. Based on these articles, a sufficient condition for obtaining a flexible, tractable and Markovian model under our model setting is

\[
\sigma_{(f,i)}(t, T) = p_n(T - t)e^{-\gamma_i(T-t)},
\]

where \(p_n\) is an \(n\)th-order polynomial function. To keep the number of parameters
manageable, we choose \( n = 1 \) which gives us

\[
\sigma_{(f,i)}(t, T) = (\alpha_{(0,i)} + \alpha_{(1,i)}(T - t))e^{-\gamma_i(T - t)}, \quad i = 1, \ldots, N.
\] (2.4)

Substituting equations (A.1) and (2.4) into the original model (2.1) gives us

\[
\begin{align*}
df(t, T) &= \sum_{i=1}^{N} v_i(t) \sigma_{(f,i)}(t, T) \int_t^T \sigma_{(f,i)}(t, u)dudt + \sum_{i=1}^{N} \sigma_{(f,i)}(t, T)\sqrt{v_i(t)}dW_i^{Q}(t), \\
dv_i(t) &= k_i(\theta_i - v_i(t))dt + \sigma_i\sqrt{v_i(t)}(\rho_i eW_i^{Q}(t) + \sqrt{1 - \rho_i^2}dZ_i^{Q}(t)),
\end{align*}
\]

where \( \sigma_{(f,i)}(t, T) \) is defined by (2.4).

**Remark 2.1.1.** When \( N = 1 \) and \( \alpha_{(1,1)} = 0 \), this model is the stochastic volatility version of Hull and White model studied by Casassus, Collin-Dufresne, and Goldstein in [12]. If we set \( \gamma_1 = 0 \), we have a stochastic volatility version of the continuous-time Ho and Lee model.

### 2.1.2 Uniqueness and Existence Theorem

Trolle and Schwartz (2008) derive a solution of this model without proving the uniqueness of the solution. Here, in this paper, we are going to prove the existence and uniqueness of the strong solution of model (2.1) under the risk-neutral measure \( Q^4 \).

**Theorem 2.1.1.** Assume that the drift coefficient \( \mu(t, T) \) and diffusion coefficient \( \sigma_{f,i}(t, T) \) in the instantaneous forward rate model satisfy the following condition

\[
\mu_{f}(t, T) = \sum_{i=1}^{N} v_i(t)\sigma_{(f,i)}(t, T) \int_t^T \sigma_{(f,i)}(t, u)du,
\] (2.5)

The solution is called a strong solution of a stochastic differential equation (SDE) if the version \( B(t) \) of Brownian motion is given in advance and the solution constructed from it is \( \mathcal{F}_t \)-adapted where \( \mathcal{F}_t \) is the \( \sigma \)-algebra generated by \( B(t) \). A weak solution consists of a probability space and a process that satisfies the SDE. There could be many weak solution but only one strong solution.
and

$$
\sigma_{(f,i)}(t,T) = (\alpha_{(0,i)} + \alpha_{(1,i)}(T - t))e^{-\gamma(T-t)}, \quad i = 1, ..., N.
$$

(2.6)

If the coefficients in the stochastic volatility model also satisfy

$$
2k_i\theta_i \geq \sigma_i^2 \quad i = 1, ..., N,
$$

(2.7)

and $\nu_i(0), \theta_i, \sigma_i$ are all positive, then, for any fixed $T > 0$, there exists a unique strong solution $f(t,T)$ adapted to the filtration $\mathcal{F}_t$ satisfying the stochastic differential equation (2.1).

The proof is presented in the online appendix.

We notice that, when we rewrite the stochastic volatility HJM model (2.1)-(2.2) in the vector style (A.4), only the drift coefficient function $\mu(t,X(t))$ can be written as a linear function of $X(t)$. The diffusion coefficient function $\sigma(t,X(t))$ can only be rewritten as a function of the components of $X(t)$. This is due to the complicated structure of the equation (2.1), especially the summation part involving all $v_i(t)$. However, this situation improves when $N = 1$.

$$
\sigma(t,X(t)) = \begin{pmatrix}
\sigma_{(f,1)}\sqrt{v_1(t)} & 0 \\
\rho_1\sigma_1\sqrt{v_1(t)} & \sqrt{1 - \rho_1^2}\sigma_1\sqrt{v_1(t)}
\end{pmatrix}
= \begin{pmatrix}
f(t,T) \\
\sqrt{v_1(t)}
\end{pmatrix}
\begin{pmatrix}
\sigma_{(f,1)} & 0 \\
\rho_1\sigma_1 & \sqrt{1 - \rho_1^2}\sigma_1
\end{pmatrix}
:= \sigma_0 X(t)\sigma_1.
$$

2.1.3 Markov Representation. The Markov representation of the model can be shown in the following proposition.
Proposition 2.1.2. (Proposition 1 in [30]) The time-t instantaneous forward interest rate with maturity $T$, $f(t,T)$, is given by

$$f(t,T) = f(0,T) + \sum_{i=1}^{N} B_{x_i}(T-t)x_i(t) + \sum_{i=1}^{N} \sum_{j=1}^{6} B_{\phi_{j,i}}(T-t)\phi_{j,i}(t),$$

(2.8)

where

$$B_{x_i}(\tau) = (\alpha_{0i} + \alpha_{1i}\tau)e^{-\gamma_i\tau},$$

$$B_{\phi_{1,i}}(\tau) = \alpha_{1i}e^{-\gamma_i\tau},$$

$$B_{\phi_{2,i}}(\tau) = \frac{\alpha_{1i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (\alpha_{0i} + \alpha_{1i}\tau)e^{-\gamma_i\tau},$$

$$B_{\phi_{3,i}}(\tau) = -\left( \frac{\alpha_{0i}\alpha_{1i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) + \frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + 2\alpha_{0i} \right) \tau + \frac{\alpha_{1i}^2}{\gamma_i^2} \right) e^{-2\gamma_i\tau},$$

$$B_{\phi_{4,i}}(\tau) = \frac{\alpha_{1i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) e^{-\gamma_i\tau},$$

$$B_{\phi_{5,i}}(\tau) = -\frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + 2\alpha_{0i} + 2\alpha_{1i}\tau \right) e^{-2\gamma_i\tau},$$

$$B_{\phi_{6,i}}(\tau) = -\frac{\alpha_{1i}^2}{\gamma_i^2} e^{-2\gamma_i\tau},$$

(2.9)

and the state variables evolve according to

$$dx_i(t) = -\gamma_i x_i(t)dt + \sqrt{v_i(t)}dW_i^Q(t),$$

$$d\phi_{1,i}(t) = (x_i(t) - \gamma_i \phi_{1,i}(t))dt,$$

$$d\phi_{2,i}(t) = (v_i(t) - \gamma_i \phi_{2,i}(t))dt,$$

$$d\phi_{3,i}(t) = (v_i(t) - 2\gamma_i \phi_{3,i}(t))dt,$$

$$d\phi_{4,i}(t) = (\phi_{2,i}(t) - \gamma_i \phi_{4,i}(t))dt,$$

$$d\phi_{5,i}(t) = (\phi_{3,i}(t) - 2\gamma_i \phi_{5,i}(t))dt,$$

$$d\phi_{6,i}(t) = (2\phi_{5,i}(t) - 2\gamma_i \phi_{6,i}(t))dt,$$

(2.10)

subject to $x_i(0) = \phi_{1,i}(0) = \ldots = \phi_{6,i}(0) = 0$.

The proof can be found in the online appendix.
Then, the time-\(t\) price of a zero-coupon bond \(P(t,T)\) with maturity \(T\), can be computed from (A.9),

\[
P(t,T) = \frac{P(0,T)}{P(0,t)} \exp \left\{ \sum_{i=1}^{N} B_{x_i}(T-t)x_i(t) + \sum_{i=1}^{6} \sum_{j=1}^{6} B_{\phi_{j,i}}(T-t)\phi_{j,i}(t) \right\},
\]

where

\[
B_{x_i}(\tau) = \frac{\alpha_{1i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (e^{-\gamma_i\tau} - 1) + \tau e^{-\gamma_i\tau},
\]
\[
B_{\phi_{1,i}}(\tau) = \frac{\alpha_{1i}}{\gamma_i} (e^{-\gamma_i\tau} - 1),
\]
\[
B_{\phi_{2,i}}(\tau) = \left( \frac{\alpha_{1i}}{\gamma_i} \right)^2 \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) \left( \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (e^{-\gamma_i\tau} - 1) + \tau e^{-\gamma_i\tau} \right),
\]
\[
B_{\phi_{3,i}}(\tau) = -\frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + \frac{\alpha_{0i}}{\gamma_i} \right) \left( \frac{\alpha_{0i}}{\gamma_i} + \frac{\alpha_{0i}}{2\alpha_{1i}} \right) (e^{-2\gamma_i\tau} - 1)
\]
\[
\quad + \left( \frac{\alpha_{1i}}{\gamma_i} + \alpha_{0i} \right) \tau e^{-2\gamma_i\tau} + \frac{\alpha_{1i}^2}{2} \tau^2 e^{-2\gamma_i\tau},
\]
\[
B_{\phi_{4,i}}(\tau) = \left( \frac{\alpha_{1i}}{\gamma_i} \right)^2 \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (e^{-\gamma_i\tau} - 1),
\]
\[
B_{\phi_{5,i}}(\tau) = -\frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + \frac{\alpha_{0i}}{\gamma_i} \right) (e^{-2\gamma_i\tau} - 1) + \alpha_{1i} \tau e^{-2\gamma_i\tau},
\]
\[
B_{\phi_{6,i}}(\tau) = -\frac{1}{2} \left( \frac{\alpha_{1i}}{\gamma_i} \right)^2 (e^{-2\gamma_i\tau} - 1).
\]

It follows that the dynamics of \(P(t,T)\) are given by

\[
\frac{dP(t,T)}{P(t,T)} = r(t)dt + \sum_{i=1}^{N} B_{x_i}(T-t)\sqrt{v_i(t)}dW_i^Q(t).
\]

### 2.2 Market Price of Risk Specifications

Before we estimate model parameters from market data, we need to transform the model from the risk-neutral measure to the real-world measure by specifying the market prices of risk \(\Lambda_{W,i}\) and \(\Lambda_{Z,i}\) defined as

\[
dW_i^P(t) = dW_i^Q(t) - \Lambda_{W,i}dt,
\]
\[
dZ_i^P(t) = dZ_i^Q(t) - \Lambda_{Z,i}dt.
\]
We follow the “extended affine” market price of risk in Cheredito, Filipovic, and Kimmel 2007 [13] by letting

\[ \Lambda W,i = \frac{\lambda_{w,i0} + \lambda_{w,ix}x_t + \lambda_{w,iv}v_i(t)}{\sqrt{v_i(t)}}, \]

\[ \Lambda Z,i = \frac{1}{\sqrt{1 - \rho_i^2}} \left[ \lambda_{Z,i0} + \lambda_{Z,iv}v_i(t) - \rho_i(\Lambda W,i \sqrt{v_i(t)}) \right]. \]

We need to make sure that, under both the risk-neutral and the real-world measure, the volatility \( v_i(t) \) cannot be zero. This can be done by requiring:

\[ 2k_i \theta_i \geq \sigma_i^2, \]

\[ 2k_i^P \theta_i^P \geq \sigma_i^2. \]

The detailed proof can be found in [13]. Such conditions ensure that the market prices of risk stay finite.

Among all the state variables in our model setting, only \( x_i \) and \( v_i \) are affected by this measure change since the other state variables \( \phi_{1,i}, \ldots, \phi_{6,i} \) are driven by deterministic equations. Then we can rewrite the stochastic equations which define the dynamics of \( x_i \) and \( v_i \) as following:

\[
dx_i(t) = -\gamma_i x_i(t) dt + \sqrt{v_i(t)} dW_i^Q(t)
= -\gamma_i x_i(t) dt + \sqrt{v_i(t)} \left[ dW_i^P(t) + \frac{\lambda_{w,i0} + \lambda_{w,ix}x_t + \lambda_{w,iv}v_i(t)}{\sqrt{v_i(t)}} \right]
= \left[ \lambda_{w,i0} + (\lambda_{w,ix} - \gamma_i) x_i + \lambda_{w,iv}v_i(t) \right] dt + \sqrt{v_i(t)} dW_i^P(t)
= (\eta^P_i + k_{x,i}^P x_i(t) + k_{iv,i}^P v_i(t)) dt + \sqrt{v_i(t)} dW_i^P(t), \tag{2.15}
\]
and

\[
d v_i(t) = k_i (\theta_i - v_i(t)) dt + \sigma_i \sqrt{v_i(t)} (\rho_i dW^Q_i(t) + \sqrt{1 - \rho_i^2} dZ^Q_i(t)) = k_i (\theta_i - v_i(t)) dt + \sigma_i \sqrt{v_i(t)} \left[ \rho_i (dW^P_i(t) + \Lambda_{W,i}) + \sqrt{1 - \rho_i^2} (dZ^P_i(t) + \Lambda_{Z,i}) \right]
\]

\[
= k_i (\theta_i - v_i(t)) dt + \sigma_i \sqrt{v_i(t)} \left[ \rho_i \left( dW^P_i(t) + \frac{\lambda_{W,i0} + \lambda_{W,ix_t} + \lambda_{W,iv} v_i(t)}{\sqrt{v_i(t)}} \right) \right.
\]

\[
+ \sqrt{1 - \rho_i^2} \left( dZ^P_i(t) + \frac{1}{\sqrt{1 - \rho_i^2}} \frac{\lambda_{Z,i0} + \lambda_{Z,iv} v_i(t) - \rho_i (\lambda_{W,i0} + \lambda_{W,ix_t} + \lambda_{W,iv} v_i(t))}{\sqrt{v_i(t)}} \right) \right]
\]

\[
= k^P_i \left( \theta^P_i - v_i(t) \right) dt + \sigma_i \sqrt{v_i(t)} \left[ \rho_i dW^P_i(t) + \sqrt{1 - \rho_i^2} dZ^P_i(t) \right] , \quad (2.17)
\]

\[
\eta^P_i = \lambda_{W,i0}, \quad k^P_{x,i} = (\lambda_{W,ix} - \gamma_i),
\]

\[
k^P_{xv,i} = \lambda_{W,iv}, \quad k^P_i = k_i - \sigma_i \lambda_{Z,iv},
\]

\[
\theta^P_i = \frac{k_i \theta_i + \sigma_i \lambda_{Z,i0}}{k^P_i}.
\]

Setting \( \lambda_{W,i0} = \lambda_{W,ix} = \lambda_{Z,i0} = 0 \) will lead to the traditional “completely affine specification” (see, e.g., [17]), while setting \( \lambda_{Z,i0} = 0 \) will produce the “essentially affine” specification (see, e.g., [18]). In contrast to these market price of risk specifications for affine models, the extended affine specification provides a better fit to US term structure data (see [12] for more details).
SECTION 3
CREDIT RISK MODEL

In this section, we introduce a credit risk model where the default intensity follows a CIR process. We explain why we choose this model and how we use it to price credit derivatives. In order to estimate the parameters of the model from market data, we also specify the market prices of risk to obtain the dynamics of the state vector under the real-world measure. Finally, we introduce different ways to simulate correlated CIR processes.

3.1 Specification of the Credit Risk Model

Let \((Ω, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0 \leq t \leq T})\) be a filtered probability space and \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the filtration associated with a standard 1-dimensional Wiener process \((B_t)_{0 \leq t \leq T}\). We assume that the default intensity process \(\lambda(t)\) is given by the following stochastic equation under the risk-neutral measure \(Q\):

\[
d\lambda(t) = k_\lambda(\theta_\lambda - \lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)} dB^Q(t), \quad \lambda(0) = \lambda_0,
\]

where \(\lambda_0, \theta_\lambda, \sigma_\lambda\) are positive constants and \(k_\lambda \in \mathbb{R}\). The following condition

\[
2k_\lambda \theta_\lambda \geq \sigma_\lambda^2,
\]

has to be imposed, so that \(\lambda(t)\) remains positive.

As Cox, Ingersoll and Ross discussed in [16], the process \(\lambda(t)\) obeys a non-central chi-squared distribution. Precisely, the density function \(P_Y\) of the random
variable $Y$ is given by

$$P_{\lambda(t)} = P_{X^2(v,h_t)/c_t}(x) = c_t P_{X^2(v,h_t)}(c_t x),$$

$$c_t = \frac{4k_\lambda}{\sigma_\lambda^2 \left(1 - \exp(-k_\lambda t)\right)},$$

$$v = 4k_\lambda \theta_\lambda / \sigma_\lambda^2,$$

$$h_t = c_t \lambda_0 \exp(-k_\lambda t),$$

where the noncentral chi-squared distribution function $X^2(\cdot, v, h)$ with $v$ degrees of freedom and non-centrality parameter $h$ has the density function defined as follows

$$P_{X^2(v,h)}(z) = \sum_{i=0}^{\infty} \frac{e^{-h/2}(h/2)^i}{i!} P_{\Gamma(i+v/2,1/2)}(z),$$

$$P_{\Gamma(i+v/2,1/2)}(z) = \frac{1}{\Gamma(i+v/2)} z^{i-1+v/2} e^{-z/2} = P_{X^2(v+2i)}(z).$$

$P_{X(v+2i)}(z)$ denotes the density of a (central) chi-squared distribution function with $v + 2i$ degrees of freedom.

There are several reasons why we choose the CIR process for the default intensity:

1. The probability $\lambda(t)$ needs to remain positive all the time.

2. Intensity models allow us to apply the results for interest rate modeling in the default model setting. With CIR processes, one can derive many closed-form formulas that are useful to price credit derivatives. For example, when we price CDS in section 4, we shall use the explicit formula (3.3)-(3.4) for the survival probability.

3. Default intensity is mean reverting which means that the expected value of the default probability tends to a long-run constant value as time goes to infinity.

Before we demonstrate the pricing formula for the survival probability which is an essential part for pricing the other credit derivatives, we introduce the following
theorem which, in contrast to the proof given by Cox, Ingersoll and Ross in their original work [16], gives another proof of the uniqueness and existence of the solution of the CIR model, from a different perspective.

**Theorem 3.1.1.** Assume that \( 2k_\lambda \theta \lambda \geq \sigma_\lambda^2 \) and \( \lambda_0, \theta_\lambda, \sigma_\lambda \) are all positive. Then, there exists a unique strong solution \( \lambda(t) \), adapted to the filtration \( \mathcal{F}_t \), satisfying the stochastic differential equation (3.1).

The proof is given in the online appendix.

It can be shown that the risk-neutral survival probability \( Q(t, T) \) is given by

\[
Q(t, T) = A(t, T)e^{-B(t, T)\lambda(t)},
\]

where

\[
A(t, T) = \left[ \frac{2\beta \exp\left\{(k_\lambda + \beta)(T - t)/2\right\}}{2\beta + (k_\lambda + \beta)(\exp\{(T - t)\beta\} - 1)} \right]^{2k_\lambda\theta_\lambda/\sigma_\lambda^2},
\]

\[
B(t, T) = \frac{2(\exp\{(T - t)\beta\} - 1)}{2\beta + (k_\lambda + \beta)(\exp\{(T - t)\beta\} - 1)},
\]

\[
\beta = \sqrt{k_\lambda^2 + 2\sigma_\lambda^2}.
\]

Under the risk-neutral measure, the dynamics for the survival probability can be easily obtained via Ito’s formula:

\[
dQ(t, T) = \lambda(t)Q(t, T)dt - B(t, T)Q(t, T)\sigma_\lambda\sqrt{\lambda(t)}dB^Q(t).
\]

### 3.2 Market Price of Risk Specification

In order to estimate the parameters from historical data, we need to do a change of measure from the risk-neutral measure \( Q \) to the real-world measure \( P \). We specify the market price of risk \( \Lambda(t) \) as follows:

\[
dW^P(t) = dW^Q(t) + \Lambda(t)dt.
\]
To preserve the same model structure under two measures, we assume that the market price of risk process $\Lambda(t)$ has the particular form

$$\Lambda(t) = \varrho \sqrt{\lambda(t)}.$$ 

which implies that the dynamics of $\lambda(t)$ under the real measure $P$ are given by

$$d\lambda(t) = (k_\lambda \theta - (k_\lambda + \varrho \sigma_\lambda)\lambda(t))dt + \sigma_\lambda \sqrt{\lambda(t)}dW^P(t), \quad \lambda(0) = \lambda_0. \quad (3.5)$$

In particular, we have

$$\frac{dQ}{dP}\bigg| = \exp\left(-\frac{1}{2} \int_0^t \varrho^2 \lambda(s)ds + \int_0^t \varrho \sqrt{\lambda(s)}dP(s)\right).$$

### 3.3 Simulation of the Correlated CIR Processes

In this section, we search for an effective method to simulate two correlated CIR processes, i.e. correlated non-central chi-squared processes. It serves as preparation for Section 6 in which we discuss how to compute counter-party risk and loss.

Assume that $x(t), y(t)$ are two correlated CIR processes

$$dx(t) = k_x(\theta - x(t))dt + \sigma_x \sqrt{x(t)}dW(t), \quad (3.6)$$

$$dy(t) = k_y(\theta - y(t))dt + \sigma_y \sqrt{y(t)}dZ(t), \quad (3.7)$$

$$dW(t)dZ(t) = \rho dt. \quad (3.8)$$

When the correlation $\rho \neq 0$, one cannot obtain the joint distribution function of two non-central chi-squared processes with which one can simulate the correlated CIR processes precisely. As a result, numerical methods become necessary. In particular, we follow an implicit Euler scheme developed by Brigo and Alfonsi [9, 8], which maintain the positivity of the CIR process, to simulate correlated CIR processes in section 6 when we need to model default events and compute counterparty risk.
SECTION 4
CREDIT DEFAULT SWAPS PRICING AND CALIBRATION

To calibrate the credit risk model, we use CDS (credit default swaps) data. We follow the technique in Brigo [9] to price single-name CDS.

4.1 Model Specification

Consider a CDS contract where the buyer agrees to pay protection rates $R_{prot}$ at times $T_{a+1},...,T_{b}$ to the seller in exchange for a single protection payment $L_{GD}$ at default time $\tau$ of an underlying entity where $T_a < \tau \leq T_b$. The $L_{GD}$ is the credit loss incurred in the event of default which equals to 1 minus the recovery rate. Then, according to the formula (5.1) in Bielecki and Rutkowski [6], the receiver CDS price at time $t$ can be written as

$$CDS(t, T_a, T_b, R_{prot}, L_{GD}) = \mathbb{E}\{D(t, \tau)(\tau - T_{\beta(\tau)} - 1)R_{prot}\mathbb{1}_{T_a < \tau < T_b} + \sum_{i=a+1}^{b} D(t, T_i)\alpha_i R_{prot}\mathbb{1}_{\tau > T_i} - \mathbb{1}_{T_a < \tau < T_b}D(t, \tau)L_{GD}|\mathcal{G}_t\},$$

(4.1)

where $T_{\beta(t)}$ is the first date among the $T_i$’s that follows $t$, and $\alpha_i = T_i - T_{i-1}$ or, more generally, $\alpha_i$ is the year fraction between $T_{i-1}$ and $T_i$. $D(t, T)$ is the discount factor at time $t$ and $\mathcal{G} = \sigma(\mathcal{F} \vee \sigma(\{\tau < u\}, u \leq t))$, $\mathcal{F}$ denotes the basic filtration without default, typically representing the information flow of interest rates, intensities and possibly other default-free market quantities while $\mathbb{E}$ denotes the risk-neutral expectation in the enlarged probability space supporting $\tau$. According to [6], this expected

---

5Here we abstract away from counter-party risk in the CDS such as that considered in Morktter, Pleus and Westerfeld [25].
value can also be written as

\[ CDS(t, T_a, T_b, R_{prot}, L_{GD}) = \frac{1_{\tau > t}}{Q(\tau > t | \mathcal{F}_t)} \mathbb{E}\left\{ D(t, \tau)(\tau - T_{\beta(\tau)-1})R_{prot}1_{T_a < \tau < T_b} \right\} \\
+ \sum_{i=a+1}^b D(t, T_i)\alpha_i R_{prot}1_{\tau > T_i} - 1_{T_a < \tau \leq T_b} D(t, \tau) L_{GD} | \mathcal{F}_t \}. \]

(4.2)

We know that the CDS spread is the fair CDS rate \( R_{prot} \) which makes the premium leg equal to the protection leg which guarantees that no cash is exchanged between the counterparties at the inception of the transaction. Then, we can derive the CDS spread formula from (4.2) as

\[ R_{a,b}(t) = \frac{L_{GD} \mathbb{E}\left\{ D(t, \tau)1_{T_a < \tau \leq T_b} | \mathcal{F}_t \right\}}{\sum_{i=a+1}^b \alpha_i Q(\tau > t | \mathcal{F}_t) \bar{P}(t, T_i) + \mathbb{E}\left\{ D(t, \tau)(\tau - T_{\beta(\tau)-1})1_{T_a < \tau \leq T_b} | \mathcal{F}_t \right\}}, \]

(4.3)

where \( \bar{P}(t, T) := \mathbb{E}[D(t, T)1_{\tau > T} | \mathcal{F}_t] / Q(\tau > t | \mathcal{F}_t) \) and

\[ \mathbb{E}[D(t, T)1_{\tau > T} | \mathcal{G}] = 1_{\tau > t} \mathbb{E}[D(t, T)1_{\tau > T} | \mathcal{F}_t] / Q(\tau > t | \mathcal{F}_t) = 1_{\tau > t} \bar{P}(t, T), \]

which is the price at time \( t \) of a defaultable bond maturing at time \( T \). In addition, let \( r_t \) be the short interest rate at time \( t \) and \( \lambda_t \) be the hazard rate. If they are independent, the CDS pricing formula (4.1) can be revised as follows:

\[ CDS(t, T_a, T_b, R_{prot}, L_{GD}) \]

(4.4)

\[ = \mathbb{E}\left\{ D(t, \tau)(\tau - T_{\beta(\tau)-1})R_{prot}1_{T_a < \tau < T_b} \right\} \\
+ \sum_{i=a+1}^b D(t, T_i)\alpha_i R_{prot}1_{\tau > T_i} - 1_{T_a < \tau \leq T_b} D(t, \tau) L_{GD} | \mathcal{G}_t \} \}

\[ = 1_{\tau > t} \left\{ R_{prot} \int_{T_a}^{T_b} \mathbb{E} \left[ \exp \left( - \int_t^u (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_t \right] du \right\} \\
+ R_{prot} \sum_{i=a+1}^b \alpha_i \mathbb{E} \left[ \exp \left( - \int_t^{T_i} (r_s + \lambda_s) ds \right) | \mathcal{F}_t \right] \\
- L_{GD} \int_{T_a}^{T_b} \mathbb{E} \left[ \exp \left( - \int_t^u (r_s + \lambda_s) ds \right) \lambda_u | \mathcal{F}_t \right] du \right\}. \]

(4.5)
If we assume zero correlation between interest rates and default intensity, we can get

\[
\mathbb{E} \left[ \exp \left( - \int_t^{T_i} (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \exp \left( - \int_t^{T_i} r_s ds \right) \mid \mathcal{F}_t \right] \mathbb{E} \left[ \exp \left( - \int_t^{T_i} \lambda_s ds \right) \mid \mathcal{F}_t \right], \quad (4.6)
\]

and

\[
\mathbb{E} \left[ \exp \left( - \int_t^u (r_s + \lambda_s) ds \right) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \exp \left( - \int_t^u r_s ds \right) \mid \mathcal{F}_t \right] \mathbb{E} \left[ \exp \left( - \int_t^u \lambda_s ds \right) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \exp \left( - \int_t^u r_s ds \right) \mid \mathcal{F}_t \right] \left( - \frac{d}{du} \mathbb{E} \left[ \exp \left( - \int_t^u \lambda_s ds \right) \mid \mathcal{F}_t \right] \right). \quad (4.7)
\]

Substituting (4.6) and (4.7) into (4.4) and (4.3), we can get

\[
CDS(t, T_a, T_b, R_{prot}, L_{GD}) = \mathbb{1}_{r > t} \left\{ R \int_{T_a}^{T_b} \mathbb{E} \left[ \exp \left( - \int_t^u r_s ds \right) \mid \mathcal{F}_t \right] \left( - \frac{d}{du} \mathbb{E} \left[ \exp \left( - \int_t^u \lambda_s ds \right) \mid \mathcal{F}_t \right] \right) \right. \\
(u - T_{\beta(u) - 1}) du + R_{prot} \sum_{i=a+1}^b \alpha_i \mathbb{E} \left[ \exp \left( - \int_t^{T_i} r_s ds \right) \mid \mathcal{F}_t \right] \mathbb{E} \left[ \exp \left( - \int_t^{T_i} \lambda_s ds \right) \mid \mathcal{F}_t \right] \\
- L_{GD} \int_{T_a}^{T_b} \mathbb{E} \left[ \exp \left( - \int_t^u r_s ds \right) \mid \mathcal{F}_t \right] \left( - \frac{d}{du} \mathbb{E} \left[ \exp \left( - \int_t^u \lambda_s ds \right) \mid \mathcal{F}_t \right] \right) du \bigg\}, \quad (4.8)
\]

and

\[
R_{a,b}(t) = L_{GD} \int_{T_a}^{T_b} \mathbb{E} \left[ \exp \left( - \int_t^{T_i} r_s ds \right) \mid \mathcal{F}_t \right] \left( - \frac{d}{du} \mathbb{E} \left[ \exp \left( - \int_t^{T_i} \lambda_s ds \right) \mid \mathcal{F}_t \right] \right) / \\
\sum_{i=a+1}^b \alpha_i \mathbb{E} \left[ \exp \left( - \int_t^{T_i} r_s ds \right) \mid \mathcal{F}_t \right] \mathbb{E} \left[ \exp \left( - \int_t^{T_i} \lambda_s ds \right) \mid \mathcal{F}_t \right] + \\
\mathbb{1}_{r > t} \int_{T_a}^{T_b} \mathbb{E} \left[ \exp \left( - \int_t^u r_s ds \right) \mid \mathcal{F}_t \right] \left( - \frac{d}{du} \mathbb{E} \left[ \exp \left( - \int_t^u \lambda_s ds \right) \mid \mathcal{F}_t \right] \right) (u - T_{\beta(u) - 1}) du \bigg\}. \quad (4.9)
\]
So, based on our model, when the interest rate and credit spread are independent, we have the closed form for the credit default spread by substituting equation (2.11) and (3.4) into the above equation.

However, valuing credit default swap with correlated interest rates and hazard rates is non-trivial, since there is no closed form formula for $\bar{P}(t, T)$ or the other terms at time $T_a$. We would thus be forced, in principle, to sub-simulate paths from $T_a$ on just to be able to obtain the underlying asset of the option at $T_a$. This is extremely computation-intensive and we need to find alternatives. One solution is to assume zero correlation between interest rate and intensity from $T_a$ on. In 2003, Brigo and Alfonsi [8] showed that the parameter $\rho$ has an impact on CDS valuation that is typically a fraction of the bid-ask spread, so that one may safely set $\rho = 0$ when pricing CDS and still calibrate the $\rho$ using Kalman filter later adding the correlation influence. Therefore, in our case, we use formula (4.9) to derive the CDS spread.
To deal with the non-linearity in our interest rate and credit risk model, we estimate it using the unscented Kalman filter (UKF) proposed by Julier and Uhlman [22]. Parameters of the models are estimated from observed yield curve data, swaption volatilities, and credit default swap spreads. All of the pricing formulas for the interest rate derivatives and credit derivatives used in the calibration can be easily found in [30] or [10].

5.1 Estimation Process of the Interest Rate Model

We start by describing the data set used in calibrating the interest rate model. Then we derive the transition function of the state variable which governs the interest rate model (in the online appendix)(2.1). Finally, we present the estimation results and the corresponding analysis.

5.1.1 Data set.

Our data set consists of weekly observations of LIBORs, swap rates and log-normal implied ATMF swaption volatility from Jun 15, 2007 to Oct 15, 2010, a particular volatile market period during the recent credit crisis. All observations are closing mid-quotes on Fridays and are obtained from Bloomberg. The LIBOR/swap term structures consist of LIBORs with maturities of 3, 6, 9 and 12 months, and swap rates with maturities 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20, 30 years. The ATMF swaptions have underlying swap maturities of 1, 2, 3, 5, 7, and 10 years and option maturities of 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, and 5 years, which is a total of 42 swaptions.

We calibrate a forward rate curve on each observation date using the following
Nelson and Siegel parametrization (for details, please see [26]):

\[ f(t, T) = \beta_0 + \beta_1 e^{-\gamma_1(T-t)} + \beta_2(T - t)e^{-\gamma_2(T-t)}. \] (5.1)

The parameters are calibrated by minimizing the mean-squared percentage differences between the observed LIBOR and swap rates and those implied ones by using forward rate curve generating by the above equation. Then, we can compute swaption prices from the lognormal pricing formulas using zero-coupon bonds computed from the calibrated forward rate curve (for details, please see [7]).

We use LIBORs, swap rates and swaption prices computed from the above procedure to calibrate model parameters. LIBORs and swap rates can be computed simply from the zero-coupon curve (see [30]) and a payer swaption can be valued as a European put option on a coupon bond by using the stochastic duration approach method developed by Wei [31] and the pricing formula of European put option developed by Trolle and Schwartz in [30].

### 5.1.2 Estimation Method and Results.

The model parameters are estimated using the unscented Kalman filter method. The online appendix describes the steps to transform the model into a state-space representation. We estimate our model for \( N = 1, 2 \) and 3. We observe from the estimation results shown in table 5.1:

- For all the models, correlation between interest rate and its volatility are negative and statistically significant and this corresponds well to the market conditions between 2007 and 2010. During the crisis, the short end of the yield curve dropped dramatically from 5% to 0.2% while volatility mostly increased.

- For all the models, the estimates of \( \alpha_{0,i}, \alpha_{1,i}, \) and \( \gamma_i \) imply that forward rate volatility functions are hump shaped which matches the findings in Amin and Morton [2]; Moraleda and Vorst [24]; Ritchken and Chuang [28]; and Mercurio and Moraleda [23].
5.2 Estimation of the Credit Risk Model

In this section, we present estimation results for the parameters in the credit risk model. We also estimate the correlation between interest rate volatility and credit default probability. We start by describing the data set. Then, using the same technique which we used to derive the transition function of the state variables of interest rate model, we can deduce the transition function of the state variables of joint interest rate model and credit model. Finally, at the end of this section, the estimation results are presented and interpreted.

5.2.1 Data set. Our data set consists of weekly observations of LIBORs, swap rates, lognormal implied ATMF swaption volatilities and credit default swap spreads of 14 companies of four industries from June 15, 2007 to September 17, 2010. All observations are closing mid-quotes on Fridays obtained from Bloomberg.

The CDS spreads have different maturities ranged from 1 year to 10 years for different reference entities. For each reference entity, we obtain the credit rating information from the Morningstar Corporate Credit Ratings list posted in the beginning of 2011. In order to assess the generality of our conclusion, the reference entities are chosen from different industries which include financial services: Bank of America (BAC), Citi Group (C), Goldman Sachs (GS), JP Morgan Chase (JPM), American Express (AXP), American International Group (AIG), Morgan Stanley (MS), Capital One Financial (COF), International Lease Finance Corp., Merrill Lynch; industrials: General Electric (GE) and United Technology (UTX); technology: IBM; energy: Exxon Mobil (XOM). The credit rating of those companies ranges from BBB to AAA. The majority of our sample are chosen from financial services since the recent credit

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6This rating list can be found from http://www.morningstar.com/credit-rating/corporate.aspx. The reason we choose Morningstar instead of credit rating from the other source is that it is open, free and easy to access online.
crisis had more impact on the financial industry. They are representative of the financial industry. The companies in the sample include three banks, three brokerages, two credit card companies, one insurance company, and one leasing company. Two industrial firms, one technology firm, and one energy firm are used for comparison purposes. The companies in the sample have CDS contracts with at least four different tenors actively traded and sufficiently large number of data points for estimation purposes. In Table 5.2, we present the summary statistics of the credit default swap spreads of each company, such as the number of available tenors, sample mean, standard deviation, of those companies during our sample period, as well as their credit ratings.

Table 5.2 shows that all the companies in the sample have at least four different tenors and a large number of CDS spread data points. We also calculate the standard deviation of all the CDS spread data. Note that those companies with higher standard deviations of the CDS spread generally have lower credit ratings.

5.2.2 Estimation Method and Result. Given the estimated interest rate model from the Section 2, we can estimate the dynamics of the hazard rate (i.e., credit model) and the correlation between interest rate model and hazard rate model using observed yield curve, lognormal implied volatility of ATMF swaptions, and credit default swap spreads. The estimated parameters are presented in Table 5.3 and interpretations of the estimation results are presented at the end of this section.

Based on the above result, we make the following observations:

- In general, higher CDS spreads imply larger values of $\theta_\lambda$. For example, Figure D.1 in the online appendix shows that, AIG and ILFC have the highest 5-years CDS spreads from 2007 to 2010 and the highest values of $\theta_\lambda$ while IBM

---

$^7$The reason why we choose 5-years CDS spread data is that it is the most liquid
and UTX have the lowest CDS spreads and the lowest values of $\theta_\lambda$. Also, our estimation results for the long-run mean are consistent with their credit ratings.

- The correlation between interest and hazard rate is negative for all companies in the sample. This means interest rate decreased as the hazard rate increased. During the crisis period, as the economy worsened and firm default risk became elevated, the federal reserve bank cut interest rate.

- Similarly, the correlation between the interest rate volatility and hazard rate is positive and significant. This indicates that, when the interest rate is volatile, i.e. the interest rate may change a lot in a short period, it is more likely for a counter-party to default.

---

tenor and contains more data points relative to other tenors.
Table 5.1. Parameter estimates by the unscented Kalman filter method: maximum likelihood estimates with outer product standard errors in parentheses. $\sigma_{\text{deriv}}$ denotes the standard deviation of interest rate derivative measurement errors.

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<th>N=2</th>
<th>N=3</th>
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<td>0.0476</td>
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Loglikelihood: -4054.55, -3496.25, -3253.57
Table 5.2. Summary statistics of credit default swap spreads

<table>
<thead>
<tr>
<th>Ticker (FS)</th>
<th># of Tenor</th>
<th># of Data</th>
<th>Mean</th>
<th>Std</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bank of America</td>
<td>7</td>
<td>1211</td>
<td>120.67</td>
<td>72.69</td>
<td>BBB</td>
</tr>
<tr>
<td>Citi Group</td>
<td>6</td>
<td>1038</td>
<td>195.04</td>
<td>148.58</td>
<td>A-</td>
</tr>
<tr>
<td>Goldman Sachs</td>
<td>7</td>
<td>861</td>
<td>180.38</td>
<td>96.37</td>
<td>BBB+</td>
</tr>
<tr>
<td>JP Morgan Chase</td>
<td>7</td>
<td>1197</td>
<td>83.30</td>
<td>40.69</td>
<td>A</td>
</tr>
<tr>
<td>American Express</td>
<td>5</td>
<td>735</td>
<td>201.52</td>
<td>162.12</td>
<td>A-</td>
</tr>
<tr>
<td>American Int’l Group</td>
<td>5</td>
<td>775</td>
<td>643.75</td>
<td>705.62</td>
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</tr>
<tr>
<td>Morgan Stanley</td>
<td>5</td>
<td>270</td>
<td>208.39</td>
<td>281.39</td>
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<tr>
<td>Capital One</td>
<td>4</td>
<td>576</td>
<td>138.93</td>
<td>105.45</td>
<td>A-</td>
</tr>
<tr>
<td>Int’l Lease Finance Corp.</td>
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<td>704</td>
<td>433.26</td>
<td>BBB-</td>
</tr>
<tr>
<td>Merrill Lynch</td>
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<td>780</td>
<td>213.31</td>
<td>111.50</td>
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</table>

<table>
<thead>
<tr>
<th>Ticker (non FS)</th>
<th># of Tenor</th>
<th># of Data</th>
<th>Mean</th>
<th>Std</th>
<th>Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Electric</td>
<td>7</td>
<td>847</td>
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<td>198.49</td>
<td>AA-</td>
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<tr>
<td>United Technology</td>
<td>7</td>
<td>637</td>
<td>57.54</td>
<td>27.57</td>
<td>A</td>
</tr>
<tr>
<td>Exxon Mobil</td>
<td>6</td>
<td>678</td>
<td>36.39</td>
<td>20.47</td>
<td>AAA</td>
</tr>
<tr>
<td>IBM</td>
<td>7</td>
<td>756</td>
<td>49.11</td>
<td>23.65</td>
<td>AA-</td>
</tr>
</tbody>
</table>
Table 5.3. Parameter estimates for the CIR credit risk model: maximum likelihood estimates. In this table, FS refers to financial service.

<table>
<thead>
<tr>
<th>FS</th>
<th>$k_\lambda$</th>
<th>$\theta_\lambda$</th>
<th>$\sigma_\lambda$</th>
<th>MVR</th>
<th>$\rho_{r,\lambda}$</th>
<th>$\rho_{v,\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIG</td>
<td>0.663</td>
<td>0.093</td>
<td>0.350</td>
<td>1.616</td>
<td>-0.535</td>
<td>0.627</td>
</tr>
<tr>
<td>BAC</td>
<td>0.261</td>
<td>0.025</td>
<td>0.075</td>
<td>0.082</td>
<td>-0.462</td>
<td>0.386</td>
</tr>
<tr>
<td>C</td>
<td>0.265</td>
<td>0.031</td>
<td>0.119</td>
<td>0.345</td>
<td>-0.548</td>
<td>0.325</td>
</tr>
<tr>
<td>GS</td>
<td>0.519</td>
<td>0.026</td>
<td>0.167</td>
<td>0.869</td>
<td>-0.676</td>
<td>0.649</td>
</tr>
<tr>
<td>JPM</td>
<td>0.306</td>
<td>0.019</td>
<td>0.058</td>
<td>0.070</td>
<td>-0.404</td>
<td>0.558</td>
</tr>
<tr>
<td>AXP</td>
<td>0.286</td>
<td>0.026</td>
<td>0.119</td>
<td>2.021</td>
<td>-0.428</td>
<td>0.576</td>
</tr>
<tr>
<td>MER</td>
<td>0.702</td>
<td>0.026</td>
<td>0.190</td>
<td>2.588</td>
<td>-0.439</td>
<td>0.544</td>
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<td>MS</td>
<td>0.553</td>
<td>0.038</td>
<td>0.205</td>
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<tr>
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<td>0.489</td>
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<td>0.116</td>
<td>0.044</td>
<td>0.569</td>
<td>-0.313</td>
<td>0.751</td>
</tr>
<tr>
<td>non FS</td>
<td>$k_\lambda$</td>
<td>$\theta_\lambda$</td>
<td>$\sigma_\lambda$</td>
<td>MVR</td>
<td>$\rho_{r,\lambda}$</td>
<td>$\rho_{v,\lambda}$</td>
</tr>
<tr>
<td>GE</td>
<td>0.475</td>
<td>0.046</td>
<td>0.211</td>
<td>0.334</td>
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<td>0.376</td>
</tr>
<tr>
<td>IBM</td>
<td>0.334</td>
<td>0.013</td>
<td>0.084</td>
<td>0.090</td>
<td>-0.530</td>
<td>0.569</td>
</tr>
<tr>
<td>UTX</td>
<td>0.524</td>
<td>0.013</td>
<td>0.113</td>
<td>0.029</td>
<td>-0.836</td>
<td>0.416</td>
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<tr>
<td>XOM</td>
<td>0.237</td>
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<td>0.067</td>
<td>0.010</td>
<td>-0.675</td>
<td>0.337</td>
</tr>
</tbody>
</table>
6.1 Introduction

In section 5, we undertook an empirical analysis of interest rate volatility and credit default swap spread, exhibiting the statistical relationships between interest rate volatility and credit risk. In this section, we apply these results in computing counter-party exposure for interest rate swaps to study the impact of the correlation between interest rate volatility and default probability on these values. These are the steps we follow:

1. Using the credit model with the parameters estimated from unscented Kalman filter in section 3, we can simulate default risk of different companies by generating the hazard rate time series $\lambda(t)$ with which, for any $\Delta t \in \mathbb{R}^+$, we can compute the probability of default for each company between $t$ and $t + \Delta t$.

2. By comparing the probability of default with a uniform distributed random number, we can simulate the default event for a given company, i.e.

   - default: $\omega \leq \lambda(t)dt$,
   - no default: $\omega > \lambda(t)dt$,

   where $\omega$ is an uniform distributed random number and $\lambda(t)dt$ is the probability of default between time $t$ and $t+dt$ conditional on no earlier defaults.

3. If default happens in the step 2, we need to compute the swap price right at the default time and two weeks later using the interest rate model introduced in section 2 and its related parameters calibrated in section 5. In this step, we assume the contract is unwound or replaced two weeks after default happens.
4. With the two swap prices we computed in step 3, we can see if the collateral can cover the loss caused by the default or not. If it could not, which means the swap price at default is bigger than the swap price two weeks later, we measure the size of the loss.

5. Then, using the Monte Carlo method (with sample size 100,000), we can repeat the above steps to compute the default probability and the loss given default for different companies with different underlying interest rate swaps.

6.2 Numerical Results

In this section, we present the numerical results for default probability and annual loss computed with correlated and independent assumptions between interest volatility and hazard rate. This allows us to study how the correlation between interest rate volatility and hazard rate influences credit exposure.

First, using the parameters estimated for the credit model presented in the Table 5.3, we can compute the default probability defined as follows

\[
\sum_{1 \leq i \leq N} \frac{1(\lambda_i(t)dt \geq \omega \text{ for any } 0 \leq t \leq M)}{N},
\]

where \(N\) is the sample size, \(M\) is the maturity of the underlying swap, \(\lambda_i(t)dt\) is the default probability of the reference entity during \([t, t+dt]\) in the ith sample path and \(\omega\) is a uniformly distributed random number. The results for all companies in our sample are shown in Table 6.2. The initial guess we used to generate hazard rate time series \(\lambda(t)\) is the predictive estimate of the hazard rate on September 18th, 2010 (three days after Lehman Brothers filed for bankruptcy) computed by the unscented Kalman filter given CIR credit model with all the parameters shown in Table 5.3. The details are given in Table 6.1.

Combining the estimation results for the interest rate model in section 5, we
Table 6.1. Parameters for simulating default events to compute default probability, which include the parameters estimated for credit risk model and the initial guess for the hazard rate on September 18th, 2010.

<table>
<thead>
<tr>
<th>Company Name</th>
<th>$\kappa_\lambda$</th>
<th>$\theta_\lambda$</th>
<th>$\sigma_\lambda$</th>
<th>$\rho_{\tau,\lambda}$</th>
<th>$\rho_{\psi,\lambda}$</th>
<th>Initial Guess of $\lambda(t)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.1190</td>
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<td>0.5769</td>
<td>0.0363</td>
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<td>0.3503</td>
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<td>0.6271</td>
<td>0.1083</td>
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<tr>
<td>BAC</td>
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<td>0.3859</td>
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</tr>
<tr>
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<td>0.1186</td>
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<td>0.3247</td>
<td>0.0592</td>
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<tr>
<td>COF</td>
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<td>0.0170</td>
<td>0.1211</td>
<td>-0.7964</td>
<td>0.4891</td>
<td>0.0805</td>
</tr>
<tr>
<td>GE</td>
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<td>0.0456</td>
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<td>-0.3761</td>
<td>0.3760</td>
<td>0.0520</td>
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<tr>
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<td>0.0076</td>
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<tr>
<td>ILFC</td>
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<td>0.1162</td>
<td>0.0044</td>
<td>-0.3132</td>
<td>0.7511</td>
<td>0.1544</td>
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<tr>
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<td>-0.4044</td>
<td>0.5576</td>
<td>0.0217</td>
</tr>
<tr>
<td>MER</td>
<td>0.7029</td>
<td>0.0256</td>
<td>0.1900</td>
<td>-0.4851</td>
<td>0.6677</td>
<td>0.1329</td>
</tr>
<tr>
<td>MS</td>
<td>0.5372</td>
<td>0.0384</td>
<td>0.2059</td>
<td>-0.2751</td>
<td>0.3149</td>
<td>0.0138</td>
</tr>
<tr>
<td>UTX</td>
<td>0.5241</td>
<td>0.0131</td>
<td>0.1133</td>
<td>-0.8355</td>
<td>0.4169</td>
<td>0.0083</td>
</tr>
<tr>
<td>XOM</td>
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<td>0.0671</td>
<td>-0.6759</td>
<td>0.3371</td>
<td>0.0151</td>
</tr>
</tbody>
</table>

can compute the loss caused by counter-party default. In order to study the influence of the correlation between interest rate volatility and credit risk on the credit loss, we compute the credit loss for two different situations. In one case, interest rate volatility is correlated with the hazard rate and in the other, the interest rate volatility is assumed to be independent with the hazard rate. By comparing the results in these two cases, we can see the difference it makes to ignore the correlation between interest rate volatility and credit risk, which is the typical industry practice.

Case 1: We assume that interest rate volatility is correlated with the credit risk, with the correlation given by our estimates in section 6. Using the Monte Carlo method, we can simulate the crisis environment and default events occurring when the simulated default intensity $\lambda$ is bigger than a uniform random number. If the collateral
is not sufficient to cover the change in mark-to-market value of the underlying swap, the difference between the collateral and mark-to-market value of the underlying swap is reported as a loss. For example, in our case, we simulated 100,000 sample paths using Monte Carlo simulation. Assume that on 40 out of the 100,000 sample paths a loss occurs due to the counter-party’s default, the expected loss we report in the Table 6.3 is calculated by dividing the sum of all 40 losses by the total sample size 100,000. The second loss statistics we presented in Table 6.4 is called expected loss given default, which is computed by dividing the sum of all 40 losses by 40.

Case 2: We assume that interest rate volatility and hazard rate are independent. We compute the default probability, expected loss and expected loss given default and present the results are presented in Tables 6.5 and 6.6.

According to the numerical results shown in Table 6.3 - 6.6, the credit losses computed with correlated interest rate volatility and hazard rate are significantly greater than the ones computed from the model that assumes zero correlation between the two. This is intuitive since our estimate for the correlation between interest rate volatility and hazard rate is always positive for all the companies. When default happens (usually caused by a high hazard rate), we are more likely to get a higher interest rate volatility which lead to larger credit losses. Therefore, the correlation between interest rate volatility and hazard rate plays an important role in pricing the credit risk and ignoring it will cause one to underestimate credit risk and miscalculate collateral. We conclude this section by providing a hypothetical example below.

Example 6.2.1. Assume that Bank AAA has a collateralized 5-year interest rate swap with Goldman Sachs Group Inc. with notional 800 million USD. As discussed in Abbate [1], we require the marked-to-market value of the swap be collateralized with a minimum threshold of 0. In the case of counter-party default, we assume it takes two weeks to unwind/replace the swap. According to our result, the probability of
Goldman's default in 5 years is 18.78%. If we assume that the interest rate volatility is independent with default risk, the expected loss conditional on Goldman’s default is $2,137,600 (0.27% of the notional). However, if we allow interest rate volatility to be correlated with counterparty credit risk, the expected loss conditional on Goldman’s default will become $2,390,400 (0.30% of the notional). Therefore, the independence assumption may lead to under-estimating the expected loss given default by $252,800.

We now turn to interpreting the significance of these results, namely an increase in expected exposure for swaps of magnitude of about 10 the example of the swap with Goldman Sachs, due to the correlation between default and interest rate volatility. Any change that significantly increases swap exposure will be consequential due to the size of the swaps market, with a gross notional of greater than US$400 trillion. This figure, however, overstates aggregate risk and exposure, as large portfolios of swaps with dealers and CCPs are subject to netting and contain many offsetting swaps. It has been estimated that the impact of netting, as indicated by Heller and Vause (2012), reduces exposure by a factor of between 20 and 50. This impact may now be greater than estimated a few years ago due to increased clearing by central counterparties, which benefit from multilateral netting. However, the significance of netting could also be somewhat reduced due to the activity of swap compression services, which organize new swap trades designed to reduce outstanding notional. Counterparty exposure calculations are important for a variety of rea-

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8As discussed in Abbate [1], market participants have gradually switched to OIS discounting from LIBOR discounting since the recent credit crisis. Such change would increase the expected loss given default for both the zero and the non-zero correlation between interest rate volatility and counter-party default case. The effect would be rather minor normally, generally being less one percent (with a typical 20 bp difference between LIBOR and OIS rate), although it might be over ten percent with an extreme of 386 bp difference as in 2008. However, the difference in the expected loss given default calculated based on zero and non-zero correlation assumptions would not change qualitatively had we switched to OIS discounting.
sons. Valuation adjustments to bank derivative portfolios, namely credit valuation adjustments (CVAs), debt valuation adjustments (DVAs) and funding valuation adjustments (FVAs), are proportional to expected exposure. For the largest derivative dealer (JP Morgan Chase), the impact of the largest of these adjustments (CVAs) on annual earnings ranged from -US$2.6 to US$1.9 billion from 2011 to 2013. A change of 10% in exposure might produce a change in annual earnings of the order of US$200 million, or about 1% of JP Morgans annual net income.  

Beyond its use in valuation adjustments, expected exposure is also used in the determination of economic and regulatory capital allocated or required for derivatives counterparty risk. High percentiles of the counterparty exposure distribution (e.g., 95th or 99th) are also computed for risk management purposes; they are used for credit limit monitoring. In this paper, we computed expected exposure statistics, but we would expect the correlation between interest rate volatility and default to have a similar impact on other statistical characteristics of the exposure distribution, such as percentiles. In our examples, we assumed a collateral threshold of zero, as has been standard in the market for the last few years. However, beginning in December 2015 (in the United States), many counterparties will be required to post substantial initial margin in their bilateral transactions (see Gregory 2014). This additional margin will dramatically reduce credit exposure and CVA. However, initial margin requirements may be determined by computing credit exposure at high percentiles with collateral thresholds set to zero. The analysis presented here is germane to the determination of these margin amounts. A 10% adjustment to exposure is not particularly large in comparison with the general level of imprecision in exposure calculations. The

---

9 Note that this figure is an adjustment for all derivatives, and not just interest rate swaps. However, interest rate swaps represent by far the most heavily traded OTC derivatives. Furthermore, the impact on exposure observed in this study for swaps should be present for derivatives in other asset classes, as in a crisis accompanying a major bank default, volatilities for most underlying assets and rates should surge.
LIBOR interest rate swap market is very liquid, and this means that swap pricing is quite precise. This is not the case for risk measures, such as credit exposure, which are not subject to direct price discovery. Exposure values will depend on the choice of underlying interest rate models and the method of their calibration. For instance, models may be calibrated using historical or implied volatilities, depending perhaps on the purpose of the measurement (valuation adjustment or limits), the philosophy of the modeler and the availability of data. These volatilities can often differ by more than 10% (proportionally, not in absolute terms), leading to differences of over 10% in exposure measurement given different calibration techniques. Exposure calculations are also sensitive to the assumed margin period of risk (MPOR), which we have set to ten days. This is a standard value assumed for OTC bilateral contracts; smaller MPORs (often five days) are used and permitted for centrally cleared contracts. Our simulation, in line with usual practice in the industry, assumed that the swaps were completely replaced at once, ten days after default. This is probably not how a default for a large swap portfolio would be managed. For instance, upon Lehman’s default, LCH Clearnet (the major swap CCP) mostly hedged Lehman’s swap portfolio over the course of two days (see Norman 2011).\footnote{A swap portfolio would contain swaps with a large variety of fixed rates and maturities, determined when the swaps were initially executed. Deep liquidity in the swap market, however, is only present for a single par fixed rate per maturity, and a set of benchmark maturities. A swap portfolio could then be hedged with these benchmark swaps (along with Eurodollar futures contracts), with the residual portfolio posing only minimal basis risk.} The residual hedged portfolios posed minimal market risk, and were auctioned off over the following several weeks. In principle, the exposure should reflect the cost of hedging and liquidating the hedged portfolio. The practice of assuming a relatively long fixed MPOR and assuming liquidation of the entire portfolio at the end of the MPOR is a simplification, but it is standard and is done out of expediency. Missing from this analysis is an attempt to incorporate the impact of illiquidity on swap exposure. The market experience
after Lehmans default suggests that barring a far more severe crisis than the one that occurred in 2008 a defaulters swap portfolio could be hedged over a few days. In this case, the MPOR that we have used would still be prudent. Illiquidity also leads to higher bid ask spreads. Normally, for the most liquid swaps, bid ask spreads quoted for swap rates are a fraction of a basis point (bp), but they did increase to 1 or 2 bps during the crisis, corresponding to a cost of several basis points in the price of a five-year swap. This is comparable to the difference we observed in exposure with and without the default interest rate volatility correlation. However, quoted bid ask spreads may understate the cost of liquidating a very large position. The results that we have presented in this paper assume sufficient liquidity in the swap market to enable hedging within a few days, as has been the case historically. If liquidity were to evaporate, in a scenario far more extreme than the 2008 crisis, losses could be far greater than forecasted by this model. This possibility could be addressed through stress testing, which would likely be rather ad hoc when involving a scenario without historical precedent, rather than attempting to extrapolate losses using the type of stochastic modeling framework presented in this paper. It is likely that alternative model specifications would lead to considerably different estimates of the impact of default and interest rate volatility correlation on swap exposure. This is because different model specifications will be characterized by qualitatively different asymptotic behavior in the dependence of default and interest rate volatility. Consider the case in which the hazard rate process and the stochastic interest rate volatility process are perfectly correlated. When a large (eg, 99.9%) shock to interest rate volatility occurs, a similar shock will be felt by the hazard rate. As the interest rate volatility shock grows, hazard rates will increase, but they will never reach the point of inducing certain default over ten days. Another choice might have been to use a structural (Merton) type model to capture default (see Bielecki and Rutkowski 11)

11Historical bid ask spread data for swaps is available from Bloomberg.
2002), with default determined when an underlying diffusive variable passes below a barrier. In this case, we would observe considerably different dependence behavior when there is a perfect correlation between the diffusive credit variable and the drive of interest rate volatility. When small shocks to volatility occur, the credit variable will not move far enough down to induce default. For a sufficiently large volatility shock, however, the credit variable will pass the barrier, and default will always occur. With perfect correlation, the two types of models will at least yield qualitatively very different distributions of volatility conditional on default, and consequently probably substantially different values for exposure.
Table 6.2. Cumulative default probability for different companies as a function of maturity

<table>
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<tr>
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Table 6.3. *Expected loss* on a collateralized interest rate swap with various maturities and counter-parties when the interest rate volatility is *correlated* with the hazard rate

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Table 6.4. *Expected loss given default* on a collateralized interest rate swap with various maturities and counter-parties when the interest rate volatility is *correlated* with the hazard rate

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Table 6.5. *Expected loss* on a collateralized interest rate swap with various maturities and counter-parties when interest rate volatility is *independent* of the hazard rate.

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Table 6.6. *Expected loss given default* on a collateralized interest rate swap with various maturities and counter-parties when interest rate volatility is *independent* of the hazard rate

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SECTION 7
CONCLUSION

In this paper, we studied the impact of the correlation between interest rate volatility and counterparty default probability on the credit risk of collateralized interest rate derivatives contracts. An interest rate model with stochastic volatility and a reduced form default model, in which the default intensity is correlated with interest rate volatility, were proposed and estimated from market data. Our results show that ignoring the correlation between interest rate volatility and a counterparty's default probability may underestimate the amount of counterparty credit risk, or the amount of initial margin needed to limit counterparty exposure. Future work might explore the impact of using other types of models, particularly for default, on these results.
APPENDIX A

PROOFS
Theorem A.0.1. (This theorem appears as Theorem 2.1 in the main text of the paper.) Assume that the drift coefficient $\mu(t,T)$ and diffusion coefficient $\sigma_{f,i}(t,T)$ in the instantaneous forward rate model satisfy the following condition

$$\mu_f(t,T) = \sum_{i=1}^{N} v_i(t) \sigma_{f,i}(t,T) \int_{t}^{T} \sigma_{f,i}(t,u)du,$$  \hspace{1cm} (A.1)

and

$$\sigma_{f,i}(t,T) = (\alpha_{(0,i)} + \alpha_{(1,i)}(T-t))e^{-\gamma_i(T-t)}, \quad i = 1, \ldots, N.$$  \hspace{1cm} (A.2)

If the coefficients in the stochastic volatility model also satisfy

$$2k_i \theta_i \geq \sigma_i^2 \quad i = 1, \ldots, N,$$  \hspace{1cm} (A.3)

and $v_i(0), \theta_i, \sigma_i$ are all positive, then, for any fixed $T > 0$, there exists a unique strong solution $f(t,T)$ adapted to the filtration $\mathcal{F}_t$ satisfying the stochastic differential equation (2.1).

Proof. Since the diffusion coefficient is given by equation (2.3), substituting in equation (2.4) gives us

$$\mu_f(t,T) = \sum_{i=1}^{N} v_i(t) \left( \alpha_{(0,i)} + \alpha_{(1,i)}(T-t) \right) e^{-\gamma_i(T-t)} \int_{t}^{T} \left( \alpha_{(0,i)} + \alpha_{(1,i)}(u-t) \right) e^{-\gamma_i(u-t)}du$$

$$= \sum_{i=1}^{N} v_i(t) \left( \alpha_{(0,i)} + \alpha_{(1,i)}(T-t) \right) e^{-\gamma_i(T-t)}$$

$$\left[ \int_{t}^{T} \alpha_{(0,i)} e^{-\gamma_i(u-t)}du + \int_{t}^{T} \alpha_{(1,i)}(u-t)e^{-\gamma_i(u-t)}du + \alpha_{(1,i)}(u-t) \right]$$

$$= \sum_{i=1}^{N} v_i(t) \left( \alpha_{(0,i)} + \alpha_{(1,i)}(T-t) \right) e^{-\gamma_i(T-t)}$$

$$\left[ -\frac{\alpha_{(0,i)}}{\gamma_i} (e^{-\gamma_i(T-t)} - 1) \left( \frac{\alpha_{(0,i)}}{\gamma_i} + \frac{\alpha_{(i,i)}}{\gamma_i^2} \right) - \frac{1}{\gamma_i} \left( \alpha_{(1,i)}(T-t)e^{-\gamma_i(T-t)} \right) \right]$$

$$:= \sum_{i=1}^{N} v_i(t) \mu_{f,i}(t,T),$$
where \( \mu_{(f,i)}(t,T) \) is defined by
\[
\mu_{(f,i)}(t,T) = \left( \alpha_{(0,i)} + \alpha_{(1,i)}(T - t) \right) e^{-\gamma_i(T-t)} \\
- \frac{\alpha_{(0,i)}}{\gamma_i} (e^{-\gamma_i(T-t)} - 1) \left( \frac{\alpha_{(0,i)}}{\gamma_i} + \frac{\alpha_{(i,i)}}{\gamma_i^2} \right) - \frac{1}{\gamma_i} \left( \alpha_{(1,i)}(T - t) e^{-\gamma_i(T-t)} \right).
\]

Let \( X(t) = [f(t,T), v_1(t), \ldots, v_N(t)]^T \). Then, according to equations (2.1) and (2.2) and the above result, \( X(t) \) is defined by following stochastic differential equation
\[
dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dB(t),
\]
where drift coefficient \( \mu(t, X(t)) : [0, T] \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{N+1} \) and diffusion coefficient \( \sigma(t, X(t)) : [0, T] \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{(N+1) \times 2N} \) are measurable functions defined by
\[
\mu(t, X(t)) = \begin{pmatrix} 0 & \mu_{(f,1)} & \mu_{(f,2)} & \cdots & \mu_{(f,N)} \\ 0 & k_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & k_N \end{pmatrix} X(t) + \begin{pmatrix} k_1 \theta_1 \\ \vdots \\ k_N \theta_N \end{pmatrix}, \quad (A.5)
\]
and
\[
\sigma(t, X(t)) = \begin{pmatrix} \sigma_{(f,1)}\sqrt{v_1(t)} & \cdots & \sigma_{(f,N)}\sqrt{v_N(t)} & \cdots \\ \rho_1\sigma_1\sqrt{v_1(t)} & \cdots & 0 & \cdots \\ \vdots & \cdots & \vdots & \ddots \\ 0 & \cdots & \rho_N\sigma_N\sqrt{v_N(t)} & \cdots \end{pmatrix} \begin{pmatrix} \sqrt{1 - \rho_1^2}\sigma_1\sqrt{v_1(t)} & \cdots & 0 \\ \vdots & \cdots & \vdots \end{pmatrix}, \quad (A.7)
\]
where $\mu_0$ and $\mu_1$ are defined by

$$
\mu_0 = \begin{pmatrix}
0 & \mu_{(f,1)} & \mu_{(f,2)} & \ldots & \mu_{(f,N)} \\
0 & k_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & k_N
\end{pmatrix}
$$

and

$$
\mu_1 = \begin{pmatrix}
0 \\
k_1 \theta_1 \\
\vdots \\
k_N \theta_N
\end{pmatrix},
$$

and $B(t) = (B_1(t), \ldots, B_{2N}(t))^T$ is a standard $2N$-dimensional Wiener process.

We notice that the coefficients $k_i, \theta_i$ and $\sigma_i$ satisfy condition (A.3). Then, according to the result presented in Cox, Ingersoll and Ross (1985), the volatility process $v_i(t), i = 1, \ldots, N$ governed by stochastic differential equation (2.2) is non-negative all of the time.
Then, for any $X(t), Y(t) \in \mathbb{R}^{N+1}$, we have

$$|\mu(t, X(t)) - \mu(t, Y(t))|^2$$

$$= \left| \begin{pmatrix} 0 & \mu(f,1) & \mu(f,2) & \cdots & \mu(f,N) \\ 0 & k_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & k_N \end{pmatrix} \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_{N+1} - y_{N+1} \end{pmatrix} \right|^2$$

$$= \left( \sum_{i=2}^{N+1} \mu(f,i-1)(x_i - y_i) \right)^2 + \sum_{i=2}^{N+1} k_{i-1}^2 (x_i - y_i)^2$$

$$\leq N \sum_{i=2}^{N+1} \mu^2_{(f,i-1)}(x_i - y_i)^2 + \sum_{i=2}^{N+1} k_{i-1}^2 (x_i - y_i)^2$$

$$\leq (N + 1) \max_{2 \leq i \leq N} (\mu^2_{(f,i-1)}, k_{i-1}^2) \sum_{i=2}^{N+1} k_{i-1}^2 (x_i - y_i)^2$$

$$= (N + 1) \max_{2 \leq i \leq N} (\mu^2_{(f,i-1)}, k_{i-1}^2) |X(t) - Y(t)|^2.$$
For the diffusion coefficient $\sigma(t, X(t))$, for any $X(t), Y(t) \in \mathbb{R}^{N+1}$, we have

$$|\sigma(t, X(t)) - \sigma(t, Y(t))|^2$$

$$= \left| \begin{array}{ccc}
\sigma_{(f,1)}(\sqrt{x_1(t)} - \sqrt{y_1(t)}) & \ldots & \sigma_{(f,N)}(\sqrt{x_N(t)} - \sqrt{y_N(t)}) \\
\rho_1\sigma_1(\sqrt{x_1(t)} - \sqrt{y_1(t)}) & \ldots & 0 \\
\vdots & \ldots & \vdots \\
0 & \ldots & \rho_N\sigma_N(\sqrt{x_N(t)} - \sqrt{y_N(t)}) \\
\sqrt{1 - \rho_1^2}\sigma_1(\sqrt{x_1(t)} - \sqrt{y_1(t)}) & \ldots & 0 \\
\vdots & \ldots & \vdots \\
0 & \ldots & \sqrt{1 - \rho_N^2}\sigma_N(\sqrt{x_N(t)} - \sqrt{y_N(t)})
\end{array} \right|^2$$

$$= \sum_{i=2}^{N+1} \left[ \sigma_{(f,i-1)}^2 \left( \sqrt{X_i(t)} - \sqrt{Y_i(t)} \right)^2 + \rho_i^2\sigma_{i-1}^2 \left( \sqrt{X_i(t)} - \sqrt{Y_i(t)} \right)^2 \\
+ (1 - \rho_i^2)\sigma_{i-1}^2 \left( \sqrt{X_i(t)} - \sqrt{Y_i(t)} \right)^2 \right]$$

$$\leq 2 \max_{1 \leq i \leq N} (\sigma_{(f,i)}^2, \sigma_i^2) \sum_{i=2}^{N+1} \left( \sqrt{X_i(t)} - \sqrt{Y_i(t)} \right)^2.$$

From the proof of Theorem 3.1, we have, for any $x, y > 0$, $|\sqrt{x} - \sqrt{y}| < |x - y|^{1/2}$. So, substituting this result into the above equation, we get

$$|\sigma(t, X(t)) - \sigma(t, Y(t))|^2 < 2 \max_{1 \leq i \leq N} (\sigma_{(f,i)}^2, \sigma_i^2) \sum_{i=2}^{N+1} |X_i(t) - Y_i(t)|.$$
Using Young’s inequality, we obtain that, for any \( N \in \mathbb{Z} \) and \( a_i \geq 0 \),
\[
\left( \sum_{i=1}^{N} a_i \right)^2 \leq N \sum_{i=1}^{N} a_i^2.
\]

Then, using this fact, we have
\[
|\sigma(t, X(t)) - \sigma(t, Y(t))|^2 < 2 \max_{1 \leq i \leq N} \left( \sigma_{(f,i)}^2, \sigma_i^2 \right) \sqrt{\sum_{i=2}^{N+1} |X_i(t) - Y_i(t)|^2}
\]
\[
= 2 \max_{1 \leq i \leq N} \left( \sigma_{(f,i)}^2, \sigma_i^2 \right) |X(t) - Y(t)|.
\]

Taking the square root of both sides, we can prove the Hölder property of the diffusion coefficient \( \sigma(t, X(t)) \), i.e.,
\[
|\sigma(t, X(t)) - \sigma(t, Y(t))| < \sqrt{2 \max_{1 \leq i \leq N} \left( \sigma_{(f,i)}^2, \sigma_i^2 \right) |X(t) - Y(t)|^2}.
\]

According to theorem in page 265 of Rogers and Williams (1990), the strong solution of equations (2.1)-(2.2) exists and is unique. \( \square \)

**Proposition A.0.2.** (Proposition 1 in Trolle and Schwarz (2009), Proposition 2.3 in the main text of the paper.) The time-t instantaneous forward interest rate with maturity \( T \), \( f(t, T) \), is given by

\[
f(t, T) = f(0, T) + \sum_{i=1}^{N} B_{x_i}(T-t)x_i(t) + \sum_{i=1}^{N} \sum_{j=1}^{6} B_{\phi_{j,i}}(T-t)\phi_{j,i}(t), \tag{A.9}
\]

where

\[
B_{x_i}(\tau) = (\alpha_{0i} + \alpha_{1i})e^{-\gamma_i \tau},
\]
\[
B_{\phi_{1,i}}(\tau) = \alpha_{1i} e^{-\gamma_i \tau},
\]
\[
B_{\phi_{2,i}}(\tau) = \frac{\alpha_{1i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) (\alpha_{0i} + \alpha_{1i} \tau) e^{-\gamma_i \tau},
\]
\[
B_{\phi_{3,i}}(\tau) = -\left( \frac{\alpha_{0i} \alpha_{1i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) + \frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + 2\alpha_{0i} \right) \tau + \frac{\alpha_{1i}^2 \tau^2}{\gamma_i^2} \right) e^{-2\gamma_i \tau},
\]
\[
B_{\phi_{4,i}}(\tau) = \frac{\alpha_{1i}^2}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0i}}{\alpha_{1i}} \right) e^{-\gamma_i \tau},
\]
\[
B_{\phi_{5,i}}(\tau) = -\frac{\alpha_{1i}}{\gamma_i} \left( \frac{\alpha_{1i}}{\gamma_i} + 2\alpha_{0i} + 2\alpha_{1i} \tau \right) e^{-2\gamma_i \tau},
\]
\[
B_{\phi_{6,i}}(\tau) = -\frac{\alpha_{1i}^2}{\gamma_i} e^{-2\gamma_i \tau}, \tag{A.10}
\]
and the state variables evolve according to

\[
\begin{align*}
    dx_i(t) &= -\gamma_i x_i(t) dt + \sqrt{v_i(t)} dW_i^Q(t), \\
    d\phi_{1,i}(t) &= (x_i(t) - \gamma_i \phi_{1,i}(t)) dt, \\
    d\phi_{2,i}(t) &= (v_i(t) - \gamma_i \phi_{2,i}(t)) dt, \\
    d\phi_{3,i}(t) &= (v_i(t) - 2\gamma_i \phi_{3,i}(t)) dt, \\
    d\phi_{4,i}(t) &= (\phi_{2,i}(t) - \gamma_i \phi_{4,i}(t)) dt, \\
    d\phi_{5,i}(t) &= (\phi_{3,i}(t) - 2\gamma_i \phi_{5,i}(t)) dt, \\
    d\phi_{6,i}(t) &= (2\phi_{5,i}(t) - 2\gamma_i \phi_{6,i}(t)) dt,
\end{align*}
\]

subject to \(x_i(0) = \phi_{1,i}(0) = \ldots = \phi_{6,i}(0) = 0\).

**Proof.** By substituting (2.4) into (A.1), we can rewrite the drift coefficient in the model (2.1) as follows

\[
\mu_f(t, T) = \sum_{i=1}^{N} v_i(t)(\alpha_{(0,i)} + \alpha_{(1,i)}(T - t)) e^{-\gamma_i(T-t)} \\
\int_t^T (\alpha_{(0,i)} + \alpha_{(1,i)}(u - t)) e^{-\gamma_i(u-t)} du \\
= \sum_{i=1}^{N} v_i(t)(\alpha_{(0,i)} + \alpha_{(1,i)}(T - t)) e^{-\gamma_i(T-t)} \\
\left[ - \left( \frac{\alpha_{0,i}}{\gamma_i} + \frac{\alpha_{1,i}}{\gamma_i^2} \right) \left( e^{-\gamma_i(T-t)} - 1 \right) - \frac{\alpha_{1,i}(T - t)}{\gamma_i} e^{-\gamma_i(T-t)} \right] \\
= \sum_{i=1}^{N} v_i(t) \left[ \frac{\alpha_{0,i}}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0,i}}{\alpha_{1,i}} \right) \left( e^{-\gamma_i(T-t)} - e^{-2\gamma_i(T-t)} \right) \\
- \frac{\alpha_{0,i}\alpha_{1,i}}{\gamma_i^2} (T - t) e^{-2\gamma_i(T-t)} - \frac{\alpha_{1,i}\alpha_{1,i}}{\gamma_i} (T - t)^2 e^{-2\gamma_i(T-t)} \\
+ \frac{\alpha_{1,i}^2}{\gamma_i} \left( \frac{1}{\gamma_i} + \frac{\alpha_{0,i}}{\alpha_{1,i}} \right) (T - t)(e^{-\gamma_i(T-t)} - e^{-2\gamma_i(T-t)}) \right].
\]
We integrate both sides of (2.1) and substitute the above result. Then, we get

\[ f(t, T) = f(0, T) + \int_0^t \mu f(s, T) ds + \sum_{i=1}^N \int_0^t \sigma_{f,i}(s, T) \sqrt{v_i(s)} dW_i^Q(s) \]

\[ = f(0, T) + \sum_{i=1}^N B_x_i(T - t) x_i(t) + \sum_{i=1}^N \sum_{j=1}^6 B_{\phi_{j,i}}(T - t) \phi_{j,i}(t), \]

where \(B_{x_i}(T - t)\) and \(B_{\phi_{j,i}}(T - t), j = 1, ..., 6\) are given by (A.10) and all the state variables are

\[ x_i(t) = \int_0^t \sqrt{v_i(s)} e^{-\gamma_i(t-s)} dW_i^Q(s), \]

\[ \phi_{1,i} = \int_0^t \sqrt{v_i(s)} (t-s) e^{-\gamma_i(t-s)} dW_i^Q(s), \]

\[ \phi_{2,i} = \int_0^t \sqrt{v_i(s)} e^{-\gamma_i(t-s)} ds, \]

\[ \phi_{3,i} = \int_0^t v_i(s) e^{-2\gamma_i(t-s)} ds, \]

\[ \phi_{4,i} = \int_0^t v_i(s)(t-s) e^{-\gamma_i(t-s)} ds, \]

\[ \phi_{5,i} = \int_0^t v_i(s)(t-s) e^{-2\gamma_i(t-s)} ds, \]

\[ \phi_{6,i} = \int_0^t v_i(s)(t-s)^2 e^{-2\gamma_i(t-s)} ds. \] (A.12)

Applying Itô’s formula to the above equations gives the dynamics shown in (A.11). \( \square \)

**Theorem A.0.3.** This theorem appears as theorem 3.1 in the main text of the paper. Assume that \(2k_\lambda \theta_\lambda \geq \sigma_\lambda^2\) and \(\lambda_0, \theta_\lambda, \sigma_\lambda\) are all positive. Then, there exists a unique strong solution \(\lambda(t)\), adapted to the filtration \(\mathcal{F}_t\), satisfying the stochastic differential equation (3.1).

**Proof.** Since coefficients \(k_\lambda, \theta_\lambda\) and \(\sigma_\lambda\) satisfy condition (3.2), according to Cox, Ingersoll and Ross (1985), the process governed by stochastic differential equation (3.1) is non-negative all the time. Then, for any \(\lambda(t) \geq 0\) and \(\Delta t > 0\), we have

\[ (\sqrt{\lambda(t) + \Delta t} - \sqrt{\lambda(t)})^2 = 2\lambda(t) + \Delta t - 2\sqrt{\lambda(t)}(\lambda(t) + \Delta t) < \Delta t, \] (A.13)
since $\lambda(t) < \sqrt{\lambda(t)(\lambda(t) + \Delta t)}$. Then, taking the square root on both sides gives us

$$|\sqrt{\lambda(t) + \Delta t} - \sqrt{\lambda(t)}| \leq \sqrt{t}.$$

So, the volatility coefficient $\sigma_{\lambda}\sqrt{\lambda(t)}$ is $\frac{1}{2}$-Holder continuous for $x \geq 0, y \geq 0$, i.e.,

$$|\sigma_{\lambda}\sqrt{x} - \sigma_{\lambda}\sqrt{y}| < \sigma_{\lambda}|x - y|^\frac{1}{2}.$$

As to the drift coefficient $k_{\lambda}(\theta_{\lambda} - \lambda(t))$, for $x \geq 0, y \geq 0$, we have

$$|k_{\lambda}(\theta_{\lambda} - x) - k_{\lambda}(\theta_{\lambda} - y)| \leq k_{\lambda}|x - y|.$$

Therefore, according to the theorem on page 265 of Rogers and Williams (1990), the strong solution exists and is unique. □
APPENDIX B
THE UNSCENTED KALMAN FILTER METHOD
B.0.1 The Unscented Kalman Filter Method for the Interest Rate Model.

In order to apply the unscented Kalman Filter method, we need to write our model in a state-space form, which consists of an observation equation (or measurement equation) and a transition equation (or state equation).

The observation equation describes the relationship between the observation \( y_t \) and the state variables \( X_t \). According to our model setting, it can be written as

\[
y_t = Z(X_t) + \eta_t, \quad \eta_t \sim N(0, V),
\]

where \( y_t \) are the time series of observations (e.g., LIBOR/swap term structure and the interest rate derivatives prices), \( X_t \) is an \( 8 \times N \) dimensional state vector, \( Z \) is the pricing function, and \( \eta_t \) is a vector of iid Gaussian distributed errors with covariance matrix \( V \). With our model specification, the state vector \( X_t \) is given by

\[
X_t = (x_1(t), \ldots, x_N(t), \phi_{1,1}(t), \ldots, \phi_{6,N}(t), v_1(t), \ldots, v_N(t))',
\]

where \( x_i(t), \ldots, x_N(t) \) and \( v_1(t), \ldots, v_N(t) \) are driven by stochastic differential equations (2.15) and (2.16). Using equations (A.11), the state vector \( X_t \) is governed by the following stochastic equation

\[
dX_t = \mu_X(t, X_t)dt + \sigma_X(t, X_t)dW(t), \quad (B.1)
\]

where \( W(t) \) is a standard \( 8N \)-dimensional independent Wiener process and the diffu-
The covariance coefficient $\sigma_X(t, X_t) : [0, T] \times \mathbb{R}^{8N} \to \mathbb{R}^{8N \times 8N}$ is defined by

$$
\sigma_X(t, X_t) = 
\begin{pmatrix}
\sqrt{v_1(t)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{v_2(t)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \sigma_1 + \sqrt{1 - \rho_1^2} \sqrt{v_1(t)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma_2 + \sqrt{1 - \rho_2^2} \sqrt{v_2(t)} & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

while the drift coefficient function $\mu_X(t, X_t) : [0, T] \times \mathbb{R}^{8N} \to \mathbb{R}^{8N}$ can be rewritten as $\mu_X(t, X_t) = b_0 + B X_t$ with $b_0 = \left( \eta^P_1, \ldots, \eta^P_N, 0, \ldots, 0, k^P_1 \theta_1, \ldots, k^P_1 \theta_N \right)^T$ and $B = \left( \begin{array}{cccccccc} 
N & k^P_{x,1} & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right)$.


So, in our model, equation (B.1) can be rewritten as

\[ dX_t = (b_0 + BX_t)dt + \sigma_X(t, X_t)dW(t). \tag{B.2} \]

Next we derive the first-order discrete dynamics (transition function) for the state vector \( X_t \).

First, integrating the left hand side of the equation (B.2) in the interval \([t, s]\), we have

\[ X_s = X_t + \int_t^s dX_u. \tag{B.3} \]

Then, taking a conditional expectation on both side gives us

\[
E_t[X_s] := E[X_s|X_t] \\
= X_t + \int_t^s E_t[dX_u] \\
= X_t + \int_t^s E_t[\mu_X(X_u)]du \\
= X_t + \int_t^s (b_0 + BE_t[X_u])du. \tag{B.7}
\]

Defining \( X(t, s) = E_t[X_s] \), the above equation can be rewritten as

\[ X(t, s) = X_t + \int_t^s (b_0 + BX(t, u))du. \tag{B.9} \]

Differentiating both sides of above equation with respect to \( s \), we have

\[ \frac{dX(t, s)}{ds} = b_0 + BX(t, s). \tag{B.10} \]

So, with the initial condition

\[ X(t, t) = X_t, \tag{B.11} \]

the first order linear differential equation (B.10) can be solved by the integrating factor method; its solution is

\[ X(t, T) = e^{B(T-t)}X_t + \int_0^{T-t} e^{Bs}b_0ds. \tag{B.12} \]
For fixed $T$, applying Itô’s lemma to above equation with respect to $t$, the dynamics of the conditional expectation is given by

$$dX(t, T) = \sigma_X(t, X_t)(e^{B(T-1)})^T dW(t). \quad (B.13)$$

Then we have

$$X_T = X(t, T) + \int_{s=t}^{T} dX(s, T) \quad (B.14)$$

$$= e^{B(T-t)}X_t + \int_{0}^{T-t} e^{Bs}b_0 ds + \int_{t}^{T} \sigma_X(s, X_s)(e^{B(T-1)})^T dW(s). \quad (B.15)$$

Then the transition equation is

$$X_T = T_t X_t + c_t + \eta_t, \quad (B.16)$$

where $T_t := e^{B(T-t)}$ is an $N \times N$ matrix, $c_t := \int_{0}^{T-t} e^{Bs}b_0 ds$ is an $N \times 1$ vector and $\eta_t := \int_{t}^{T} \sigma_X(s, X_s)(e^{B(T-1)})^T dW(t)$ is an $N \times 1$ vector of uncorrelated disturbances with mean 0 and covariance matrix $H_t = \int_{t}^{T} \sigma_X(e^{B(T-t)})^T(\sigma_X(e^{B(T-t)})^T) dt$.

The zero coupon price in (2.11) is time-inhomogeneous. To simplify the estimation process, we introduce another parameter named $\varphi$ and replace $\frac{P(0,T)}{P(0,t)}$ with the function $\exp\{-\varphi(T - t)\}$. The parameter $\varphi$ is can be interpreted as the infinite-maturity forward rate. This parameterization not only helps us to simplify the estimation process of the initial yield curve, but also allows us to express the price of the zero-coupon bond as a time-homogeneous process which will help us to estimate the other parameters later.

Now, with the observation function and the transition function we derived above, we can apply the unscented Kalman filter to calibrate all of the parameters in our model. To complete the discussion of the calibration, there are still a couple of numerical issues to address:

1. The log-likelihood function derived by the Kalman filter method is maximized by the simulated annealing method described in Kirkpatrick, Gelatt and Vecchi.
which is designed to find a good approximation of the global maximum point of a given function in a large domain. The optimal result we report is attained using several different plausible initial guesses and, compared with the results run from other reasonable initial guesses, provides a larger maximum likelihood function value.

2. We used standard fourth-order Runge-Kutta algorithm to solve the ordinary differential equations and the Gauss-Legendre quadrature formula to evaluate the integral involved in swaption calculations. For more details, please check equations (31)-(34) in Trolle and Schwartz (2009).

B.0.2 The Unscented Kalman Filter Method for the Joint Interest Rate and Credit Model. In order to use unscented Kalman filter to estimate the parameters of interest rate process and hazard rate process together, we need to rewrite their joint dynamics into state-space form, which consists of a measurement equation and a transition equation. Since the pricing formula of CDS spread not only involves the instantaneous short rate but also the hazard rate, the state vector we introduce here contains more elements than does the state vector used in the interest rate model. The measurement equation can be written as

\[ y_t = CDS(X_t) + u_t, \quad u_t \sim \mathcal{N}(0, S), \]

where \( y_t \) are the time series of CDS spreads, \( CDS \) is the pricing function for the credit default swap spread defined by (4.8), and \( u_t \) is a vector of independent and identical Gaussian measurement error with mean 0 and covariance matrix \( S \). To reduce the number of parameters in \( S \), we assume that the measurement errors are cross-sectionally independent, implying that \( S \) is diagonal. The instantaneous forward
rate is governed by

$$\begin{align*}
df(t, T) &= \mu_f(t, T) dt + \sigma_{(f,1)}(t, T) \sqrt{v(t)} dW^Q(t), \\
dv(t) &= k(\theta - v(t)) dt + \sigma \sqrt{v(t)} dZ^Q(t),
\end{align*}$$

where the drift coefficient \(\mu_f(t, T)\) and diffusion coefficient \(\sigma_{(f,1)}(t, T)\) are given by equations (A.1) and (2.4) and, the hazard rate is given by

$$d\lambda(t) = k_\lambda(\theta - \lambda(t)) dt + \sigma_\lambda \sqrt{\lambda(t)} dB^Q(t),$$

where \(W^Q(t), Z^Q(t)\) and \(B^Q(t)\) are correlated standard Wiener Processes satisfying the following conditions

$$dW^Q(t) dZ^Q(t) = \rho_{f\alpha} dt \quad dW^Q(t) dB^Q(t) = \rho_{f\lambda} dt \quad dB^Q(t) dZ^Q(t) = \rho_{\lambda\lambda} dt.$$

According to the zero-coupon bond formula (2.11) and the survival probability function (3.3), the state variable \(X_t\) of the joint models is given by

$$X_t = (x(t), \phi_1(t), \ldots, \phi_6(t), v(t), \lambda(t))'.$$

where \(\phi_1(t), \ldots, \phi_6(t)\) are defined by (A.12) and \(x(t), v(t)\) and \(\lambda(t)\) are driven by stochastic different equations (2.15), (2.16) and (3.5). Then, using the same technique we derive the transition function of the interest rate model in section 2, the transition function of \(X(t)\) is follows

$$X(T) = e^{B(T-t)X(t)} + \int_0^{T-t} e^{B(T-t)} b_0 + \int_{s=t}^T \sigma_X e^{B(T-t)} dW_t,$$

where \(b_0 = (\eta, 0, \ldots, 0, k_P \theta_P, k_\lambda \theta_\lambda)^T\), \(W(t) = (W_1(t), \ldots, W_9(t))^T\) is an 9 \times 1 inde-
pendent standard Wiener Process vector,

\[
B = \begin{pmatrix}
K_x & 0 & 0 & 0 & 0 & 0 & 0 & K_{xv} & 0 \\
1 & -\gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -2\gamma & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -\gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -2\gamma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -2\gamma & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k^P & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -k_{\lambda}
\end{pmatrix}, \quad (B.22)
\]

and

\[
\begin{pmatrix}
\sqrt{v(t)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma \rho_{f_v} \sqrt{v(t)} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_{a \lambda} - \rho_{f_v} \rho_{v \lambda}}{\sqrt{1-\rho_{f_v}^2}} & \sqrt{1-\rho_{f_v}^2} & \frac{(\rho_{a \lambda} - \rho_{f_v} \rho_{v \lambda})^2}{1-\rho_{f_v}^2} \\
\rho_{f \lambda} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_{a \lambda} - \rho_{f_v} \rho_{v \lambda}}{\sqrt{1-\rho_{f_v}^2}} & \sqrt{1-\rho_{f_v}^2} & \frac{(\rho_{a \lambda} - \rho_{f_v} \rho_{v \lambda})^2}{1-\rho_{f_v}^2}
\end{pmatrix}. \quad (B.23)
\]

With the above measurement equation and transition equation, we can calibrate the parameters involved in the credit model by using market CDS spreads for various companies.
APPENDIX C
REFERENCES APPEARING IN THE APPENDICES


APPENDIX D

FIGURES
Figure D.1. CDS spread for all 14 companies
BIBLIOGRAPHY


